Graph of Rational Functions

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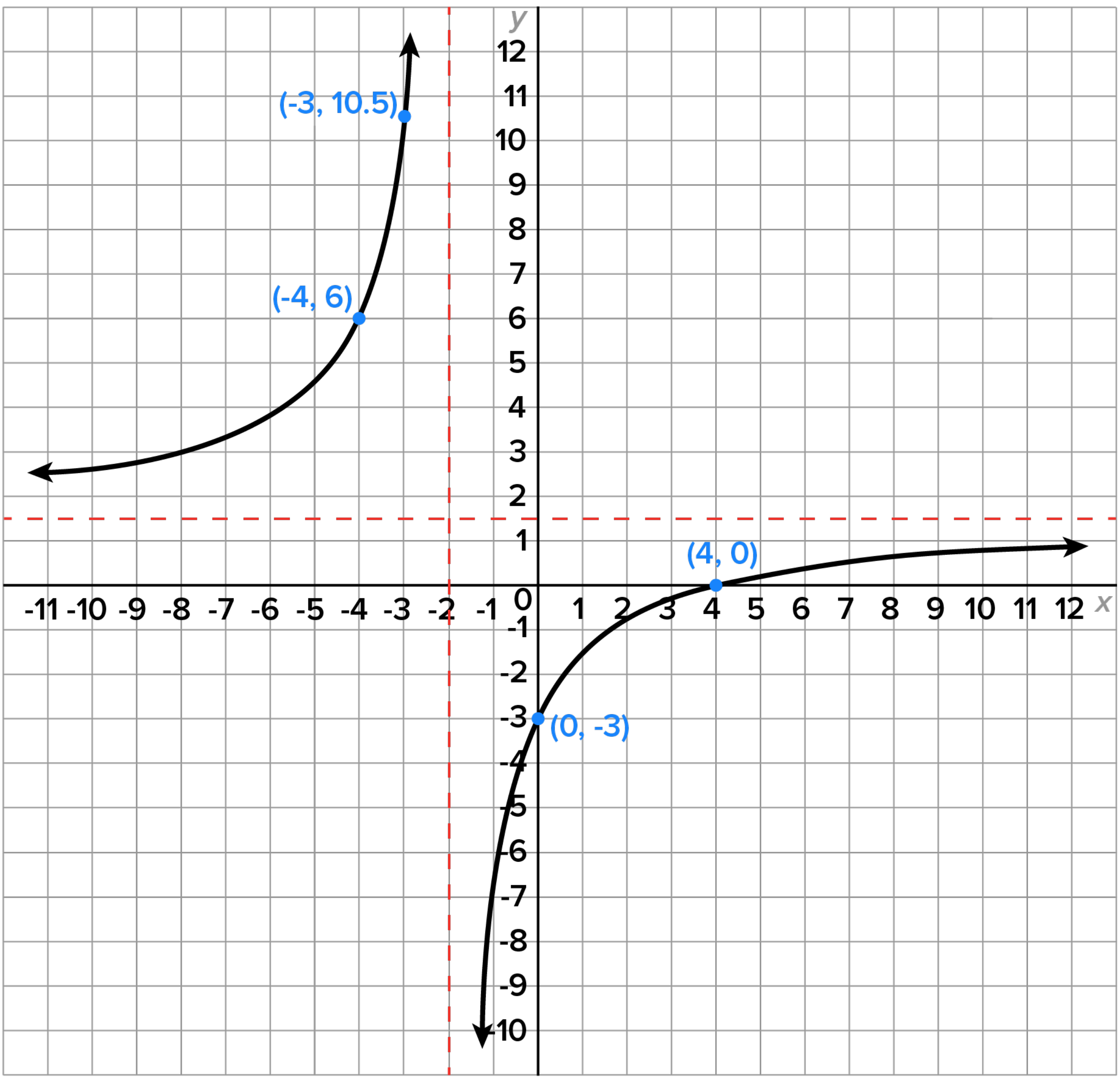
WHAT'S COVERED

In this lesson, you will learn how to determine a vertical, horizontal, or oblique asymptote of a rational function. Specifically, this lesson will cover:

1. Asymptotes

The graphs of rational functions have **asymptotes**, which are lines that aren't part of the curve itself but dictate certain behavior about the curve. The main characteristic of asymptotes is that the curve continues to approach the asymptote at the extreme ends of the graph.



Let's explore three different types of asymptotes on the graphs of rational functions.

TERM TO KNOW

Asymptote

A line that a curve approaches; the distance between the curve and an asymptote approaches zero.

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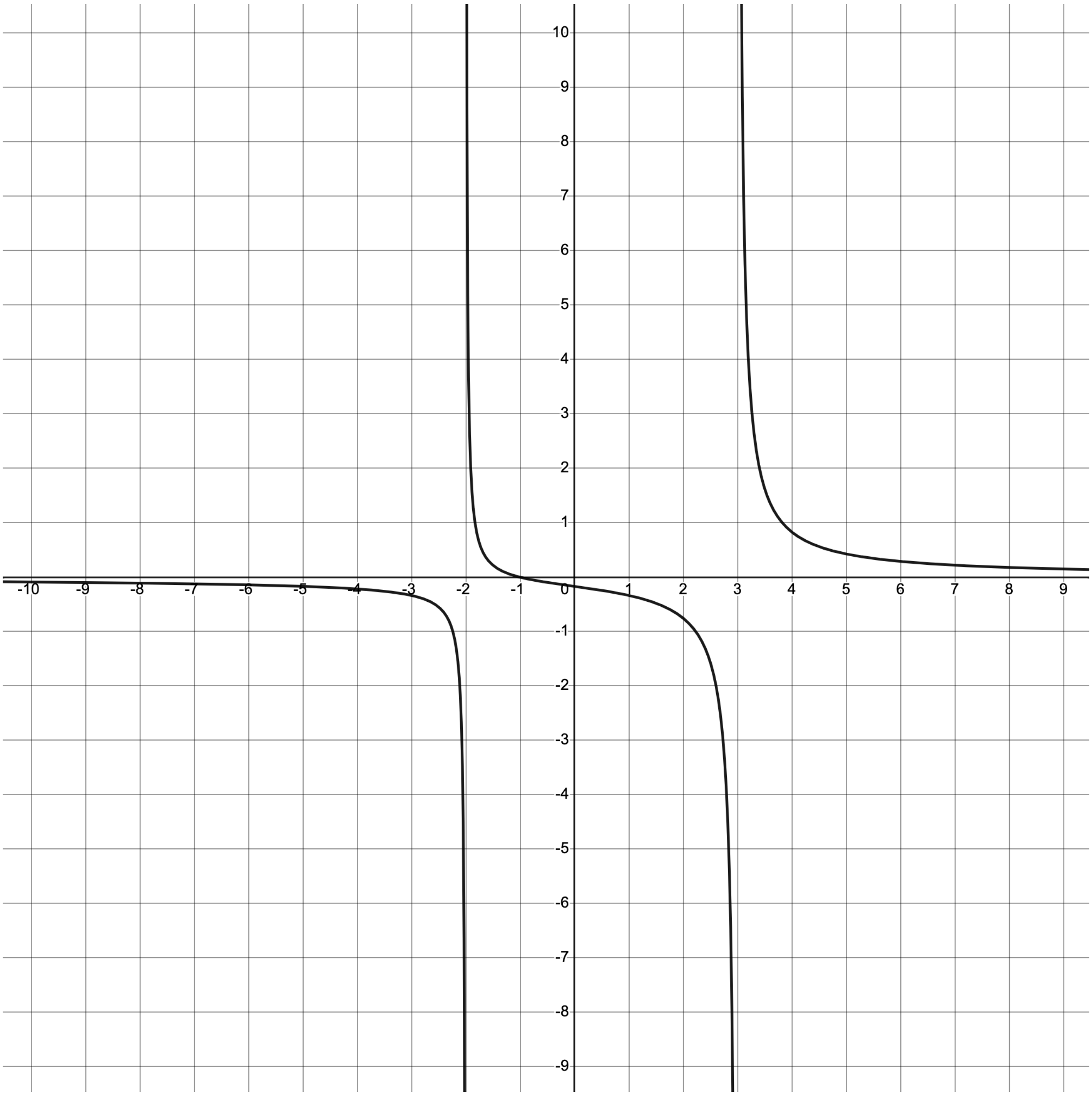
2. Vertical Asymptotes

One type of asymptote is a **vertical asymptote**, which is a vertical line that a curve approaches from the left or right.

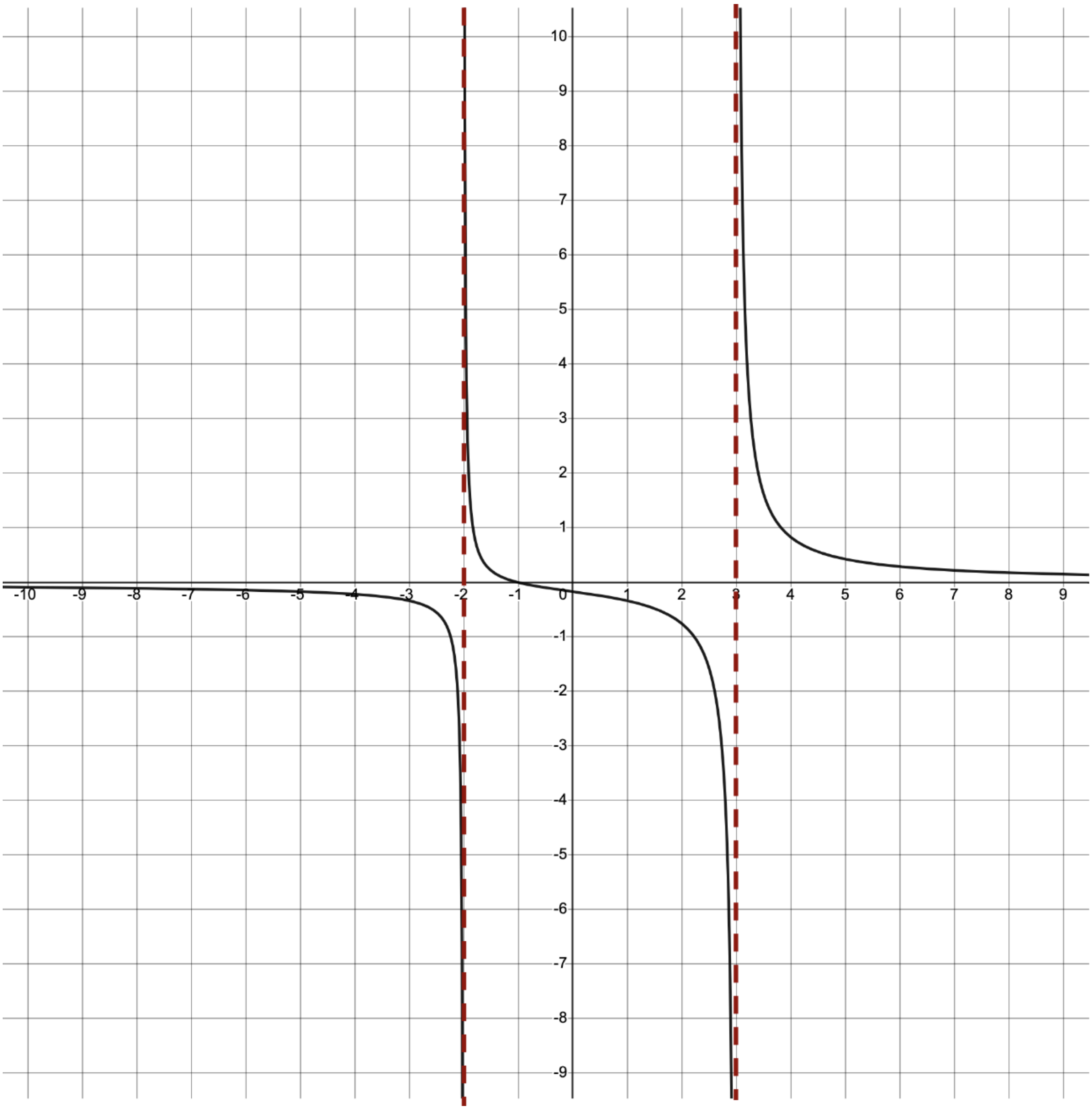
EXAMPLE Below is the graph of $f(x) = \dfrac{x+1}{x^2-x-6}$.



To find vertical asymptotes, we consider x-values that make the denominator equal to zero. These represent values for which the function is undefined since we cannot divide by zero. Finding vertical asymptotes vertically, we set the denominator equal to zero and solve for *x*:

$f(x)=\dfrac{x+1}{x^2-x-6}$	Set denominator equal to zero
$x^2-x-6=0$	Factor the quadratic expression
$(x+2)(x-3)=0$	Set each factor equal to zero
$x+2=0,\ x-3=0$	Solve each *x* in each factor
$x=-2,\ x=3$	Our solutions

Here is the same graph as above, but with the vertical asymptotes $x=-2$ and $x=3$ drawn in as dashed vertical lines.



Before we can conclude that these are vertical asymptotes, we must also make sure that these x-values do not make the numerator equal zero as well. If they do, then we have holes on the graph at these points, rather than vertical asymptotes.

Using the numerator $x+1$, plug in $x=-2$ and $x=3$.

$x+1$	Plug in $x=-2$
$(-2)+1=-1$	Plug in $x=3$
$(3)+1=4$	Both result in a non-zero numerator

We have confirmed vertical asymptotes at $x=-2$ and $x=3$.

TERM TO KNOW

Vertical Asymptote

A vertical line that a curve approaches from the left or right; the curve tends towards positive or negative infinity, and its distance to the vertical line tends towards zero.

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3. Horizontal Asymptotes

Another type of asymptote is a **horizontal asymptote**, which is a horizontal line that a curve approaches from above or below. To find horizontal asymptotes of a rational function, we must analyze the degrees of the polynomials in the numerator and denominator of the fraction.

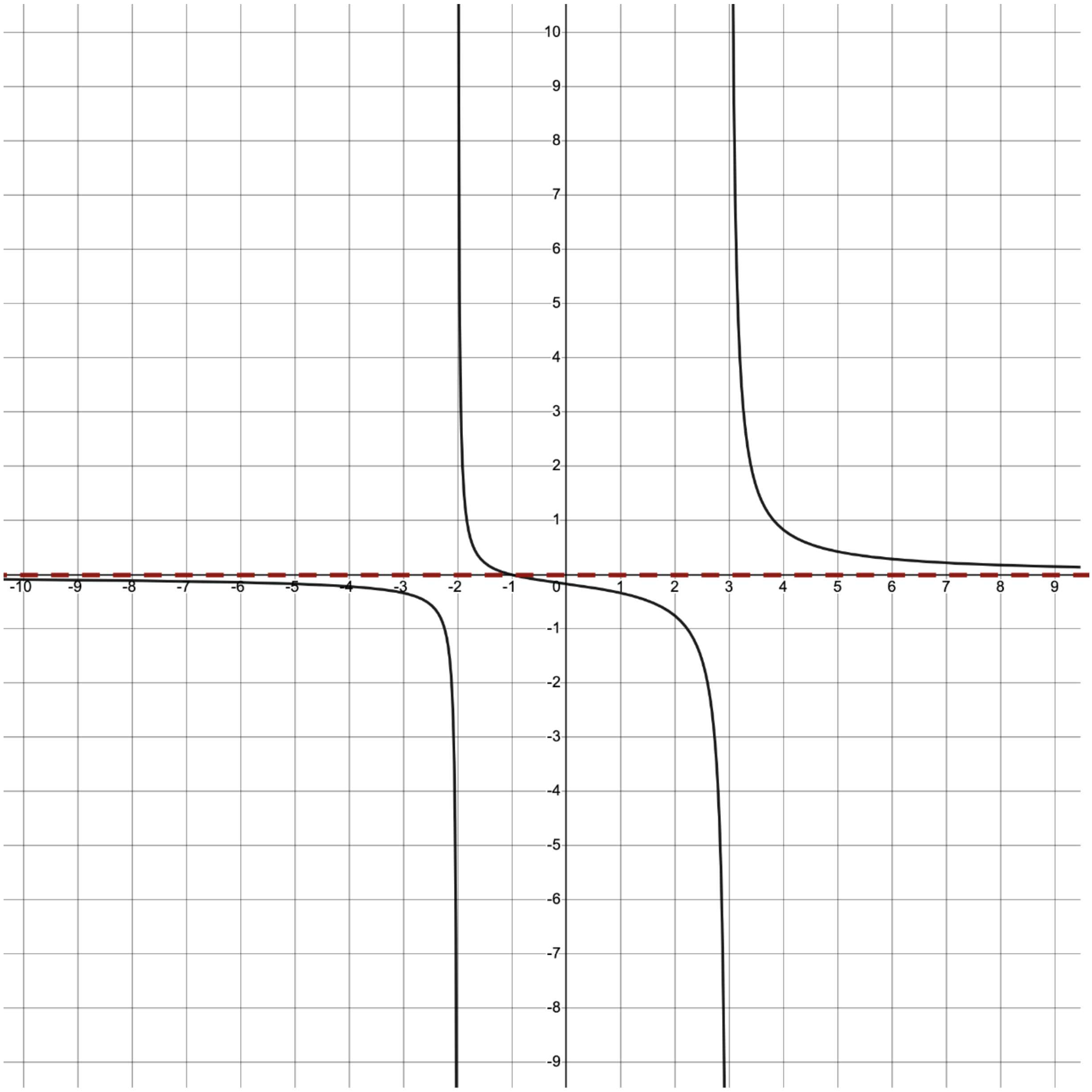
In general, we can say:

- A rational function can be expressed as $f(x) = \frac{p(x)}{q(x)}$
- The degree of $p(x)$ is m.
- The degree of $q(x)$ is n.

Comparing the degrees of $p(x)$ and $q(x)$ can help us determine the equation to horizontal asymptotes:

- If the degree of $p(x)$ is less than the degree of $q(x)$, in other words, if $m < n$, then the horizontal asymptote is the line $y = 0$.
- If $p(x)$ and $q(x)$ have the same degree, in other words, if $m = n$, then the horizontal asymptote is the line $y = \frac{a}{b}$, where a and b are the leading coefficients of the polynomials (we'll get to this later).

> ↱ EXAMPLE Let's consider the previous rational function, as it also has a horizontal asymptote. The equation, once again, is $f(x) = \frac{x+1}{x^2-x-6}$.
>
> The degree on the numerator is 1 because we only have an x, and the degree of the denominator is 2 because there is an x^2. Since the numerator has a smaller degree, the horizontal asymptote is the line $y = 0$. Here is the graph of the equation once more:



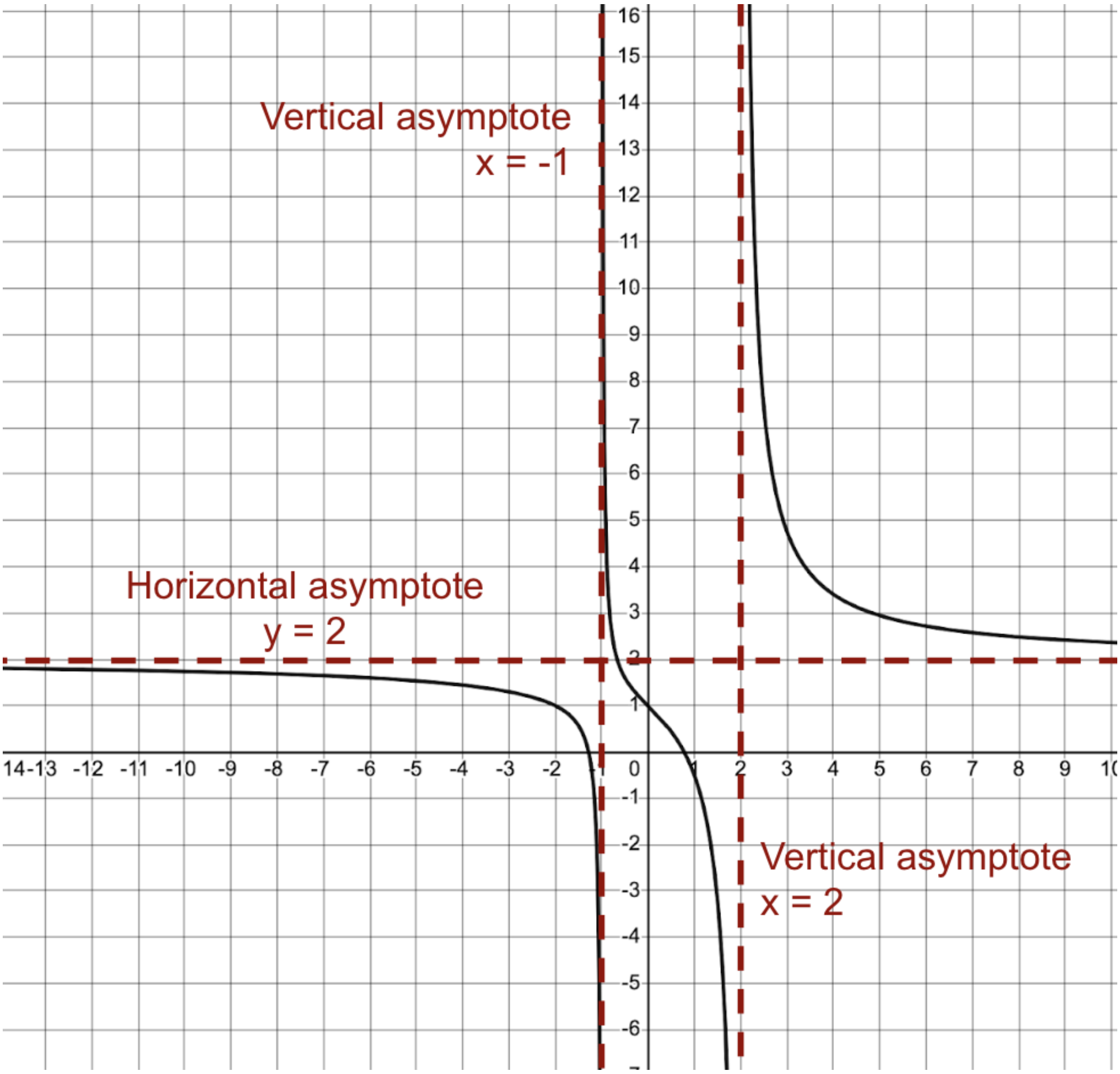
We can see that the line $y=0$, or the x-axis of the graph, is the horizontal asymptote: the curve tends towards this horizontal line as *x* approaches positive and negative infinity.

Next, let's consider the case where the degrees are equal. In this case, the horizontal asymptote is found by dividing the leading coefficients of the polynomials. The leading coefficient is the coefficient of the highest degree term in the polynomial.

→ EXAMPLE Consider the function $f(x)=\frac{2x^2+x-2}{x^2-x-2}$.

The highest term in both the numerator and denominator is x^2 , so they are both second-degree polynomials. This means that we divide the leading coefficients to find the horizontal asymptote. The leading coefficients are 2 (from the $2x^2$ in the numerator) and 1 (from the x^2 in the denominator). Therefore, the horizontal asymptote is the line $y=\frac{2}{1}$, which is just $y=2$.

Here is a sketch of the function.



We can see that there are some vertical asymptotes as well, but we want to focus here on the line $y=2$ as the horizontal asymptote.

TERM TO KNOW

Horizontal Asymptote

A horizontal line that a curve approaches from above or below; the curve tends towards a constant value, and its distance to the horizontal line tends towards zero.

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4. Oblique Asymptotes

With horizontal asymptotes, we considered either when the degrees of the polynomials in the numerator and denominator are equal to each other, or if the degree in the denominator is greater. If the degree in the numerator is greater, then the rational function has an **oblique asymptote**.

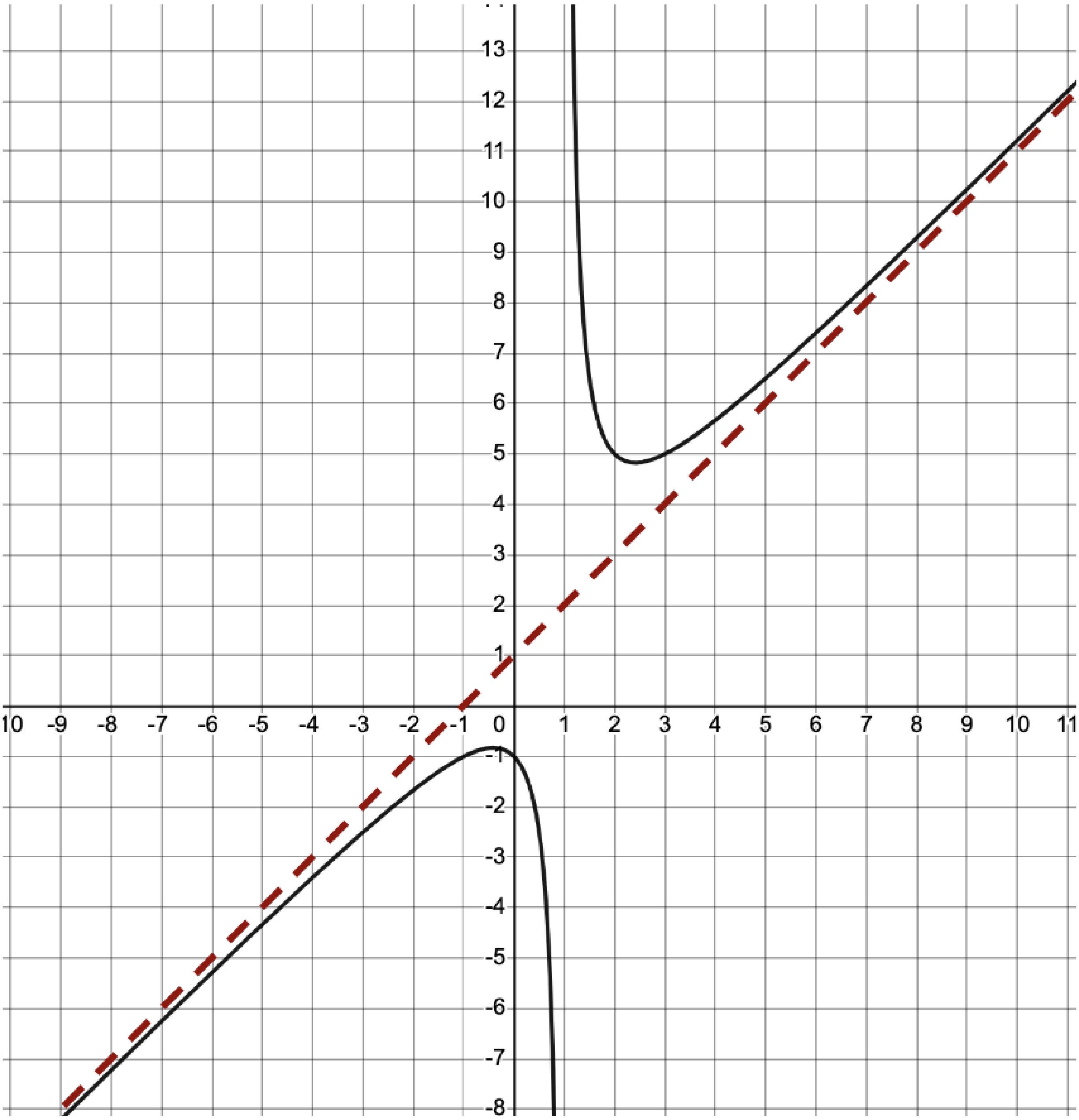
HINT

When studying oblique asymptotes, more often than not, the degree in the numerator will only be one more than the degree in the denominator. In these cases, the oblique asymptote is a linear equation in the form $y = mx + b$. If the degree in the numerator is greater by more than one degree, the equation for the oblique asymptote will not be linear.

↪ EXAMPLE Consider the function $f(x) = \frac{x^2 + 1}{x - 1}$.

The degree of the numerator is 2, while the degree of the denominator is 1, so this function will have an oblique asymptote.

Here is a sketch of the function.



Notice that the asymptote is neither horizontal nor vertical. To find the exact equation for an oblique asymptote, we do so through polynomial division. In dividing the numerator by the denominator, we take only the polynomial portion of the quotient as the equation to the oblique asymptote. Any remainder is not included in the equation.

TERM TO KNOW

Oblique Asymptote

Also called slant asymptote, a line that a curve approaches as x tends towards positive or negative infinity; it can be defined by the line $y = mx + b$, where $m \neq 0$.

SUMMARY

The graph of a rational equation have **asymptotes**, which dictates certain behavior about the curve. The graph of a rational equation has **vertical asymptotes** at the x-values that make the denominator equal to 0, provided the x-value does not make the numerator equal to 0 as well. The graph of a rational equation has a **horizontal asymptote** if the degree in the numerator is less than or equal to the degree in the denominator. The graph of a rational equation has an **oblique asymptote** if the degree in the numerator is one greater than the degree in the denominator.

TERMS TO KNOW

Asymptote

A line that a curve approaches; the distance between the curve and an asymptote approaches zero.

Horizontal Asymptote

A horizontal line that a curve approaches from above or below; the curve tends towards a constant value, and its distance to the horizontal line tends towards zero.

Oblique Asymptote

Also called slant asymptotes, a line that a curve approaches as *x* tends towards ± inifinity; it can be defined by y=mx+b, where m≠0.

Vertical Asymptote

A vertical line that a curve approaches from the left or right; the curve tends towards positive or negative infinity, and its distance to the vertical line tends towards zero.