Solving Quadratic Inequalities Representing Motion

by Sophia

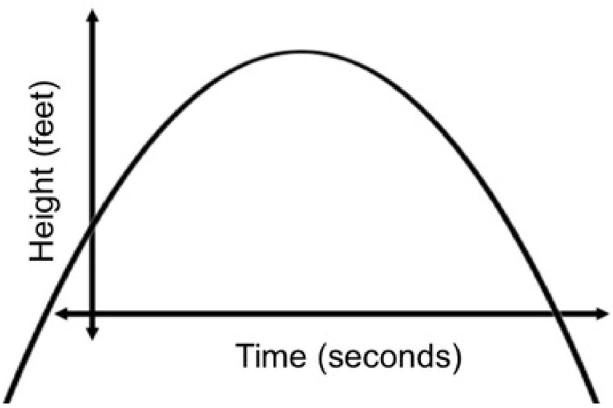
WHAT'S COVERED

In this lesson, you will learn how to determine the solution set of a quadratic inequality in a given scenario. Specifically, this lesson will cover:

1. Solving a Quadratic Inequality for an Object in Motion

Quadratic equations can be used to represent the path of objects as they go up in the air, and come back down due to the force of gravity.

EXAMPLE Imagine a ball being thrown straight up into the air. For the first few seconds, the height of the ball will be increasing, but eventually, it will reach a maximum height and then begin to fall back down, eventually hitting the group. In relation to time, the graph might look something like this:



The graph of this curve can be represented by the following equation:

$$y = -16x^2 + 80x + 6$$

DID YOU KNOW

The coefficients in this equation have important meanings within this context. $-16x^2$ comes from the force of gravity on Earth, $80x$ represents the velocity of the ball, and 6 represents the height of the player hitting the ball.

We can use this equation to answer some questions about the height of the ball, and how long it is in the air.

EXAMPLE Suppose we want to figure out when the ball is more than 70 feet in the air. We will use a quadratic inequality to represent this situation.

$$70 < -16x^2 + 80x + 6$$

We use the < symbol to show that the height of the ball (represented by the quadratic equation) has a greater value than 70. We could have alternatively written our inequality with 70 on the right side, where the inequality symbol would be >, $-16x^2 + 80x + 6 > 70$.

To solve quadratic inequalities:

1. Write as an equation set equal to zero.
2. Solve the equation (using factoring, completing the square, the quadratic formula, etc.).
3. Use solutions to create intervals on a number line.
4. Identify intervals that make the inequality statement true.

Our first step is to write the inequality as an equation set equal to zero. To do this, we'll need to move the constant term to the other side:

$70 < -16x^2 + 80x + 6$	Write as an equation
$70 = -16x^2 + 80x + 6$	Set equal to zero by subtracting 70 from both sides
$0 = -16x^2 + 80x - 64$	Our equation

To solve this equation, we can use any method, although factoring and the quadratic formula are the most common. Factoring tends to involve fewer calculations, but it can take too long to identify such integers, so it is probably best to use the quadratic formula. We are going to solve this by trying both methods, factoring and using the quadratic formula. First, let's factor out a common factor between all coefficients.

$0 = -16x^2 + 80x - 64$	Factor out -16
$0 = -16(x^2 - 5x + 4)$	Factor the quadratic expression
$0 = -16(x-4)(x-1)$	Set each factor equal to zero
$0 = x - 4, \quad 0 = x - 1$	Simplify each factor
$x = 1,\ x = 4$	Our solutions

It was easy to factor this particular equation, but that's not always the case. If it looks complicated, you can always use the quadratic formula. Let's show how to find the same answers using this formula.

If we wanted to use the quadratic formula, we would get the same answers.

$0 = -16x^2 + 80x - 64$	Identify coefficients a, b, and c
$a = -26, b = 80, c = -64$	Use quadratic formula
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Plug in $a = -26, b = 80, c = -64$
$x = \frac{-80 \pm \sqrt{80^2 - 4(-16)(-64)}}{2(-16)}$	Square 80 and multiply 4, -16, and -64
$x = \frac{-80 \pm \sqrt{6400 - 4096}}{2(-16)}$	Simplify discriminant
$x = \frac{-80 \pm \sqrt{2304}}{2(-16)}$	Evaluate square root
$x = \frac{-80 \pm 48}{2(-16)}$	Evaluate denominator
$x = \frac{-80 \pm 48}{-32}$	Separate into two fractions, one with addition and one with subtraction
$x = \frac{-80 - 48}{-32}, x = \frac{-80 + 48}{-32}$	Evaluate numerators of both fractions
$x = \frac{-128}{-32}, x = \frac{-32}{-32}$	Simplify
$x = 4$, and $x = 1$	Our solutions

We use these two solutions to form intervals on the number line. Since we have two solutions, this breaks the number line into three sections:

Next, we choose a value that lies within each interval and plug that value into the inequality, to see if it yields a true statement. It is helpful to use the inequality with zero on the one side, so we can simply compare the values to zero. We can rewrite $70 < -16x^2 + 80x + 6$ as $0 < -16x^2 + 80x - 64$.

Using this rewritten inequality, choose test values within the three intervals. We need a test value that is less than 1, a test value that is between 1 and 4, and a test value greater than 5. Let's use 0, 2, and 5 and plug them into the inequality

Test Value	$0 < -16x^2 + 80x - 64$	Result	Interval

0	$0 < -16(0)^2 + 80(0) - 64$ $0 < 0 + 0 - 64$ $0 < -64$	This inequality is FALSE, so this interval is NOT part of the solution.	$x \leq 1$
2	$0 < -16(2)^2 + 80(2) - 64$ $0 < -16(4) + 160 - 64$ $0 < -64 + 160 - 64$ $0 < 32$	This inequality is TRUE, so this interval is part of the solution.	$0 < x < 4$
5	$0 < -16(5)^2 + 80(5) - 64$ $0 < -16(25) + 400 - 64$ $0 < -400 + 400 - 64$ $0 < -64$	This inequality is FALSE, so this interval is NOT part of the solution.	$x \geq 4$

This means that the ball is above 70 feet between 1 and 4 seconds after being launched into the air.

HINT

You might have preferred to use the inequality written in factored form $0 < -16(x-4)(x-1)$, especially if the other side of the inequality is zero. This is because you can simply analyze the signs of each factor (positive or negative) to determine if the value of the quadratic will be positive or negative. From there, you can easily compare to zero without actually calculating exact values. Either way, you would get the same answers.

---

2. Quadratic Inequalities with No Solution

Let's take a look at an example where a quadratic inequality has no solutions.

↱ EXAMPLE We would like to know when the ball is above 120 feet. To answer this question, we follow the same steps as in our previous example, but with a different inequality to represent this new scenario.

$$120 < -16x^2 + 80x + 6$$

The first step is writing this inequality as an equation set equal to zero.

$120 < -16x^2 + 80x + 6$	Write as an equation
$120 = -16x^2 + 80x + 6$	Set equal to zero by subtracting 120 from both sides
$0 = -16x^2 + 80x - 114$	Our equation

Then we must solve $0 = -16x^2 + 80x - 114$. We will solve this using the quadratic formula.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Plug in $a = -26,\ b = 80,\ c = -114$

$$x=\frac{-80\pm\sqrt{80^2-4(-16)(-114)}}{2(-16)}$$ Square 80 and multiply 4, -16, and -114

$$x=\frac{-80\pm\sqrt{6400-7296}}{2(-16)}$$ Evaluate discriminant

$$x=\frac{-80\pm\sqrt{-896}}{2(-16)}$$ Non-real solution

We cannot work any further, because we have a negative value underneath the square root. There is a special term for this value when working with the quadratic formula. We call this the discriminant. If the discriminant is negative, there is no real solution to the equation, because every real number square leads to a positive value.

Since there is no real solution, we know that the ball will never even reach 120 feet in the air.

SUMMARY

A quadratic equation can be used to model the path of an object rising and falling due to gravity. If x represents time and y represents height, we can **solve a quadratic inequality for an object in motion**, such as the time interval that an object is greater than or less than a certain height. There are also situations of **quadratic inequalities with no solution**. This is when the discriminant is negative.