Solving Rational Equations

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WHAT'S COVERED

In this lesson, you will learn how to determine an equivalent rational expression by finding the least common denominator. Specifically, this lesson will cover:

1. Rational Equations and Extraneous Solutions

A **rational equation** is an equation that contains at least one rational expression. Recall that a rational expression can also be referred to as an algebraic fraction. In other words, one of the terms has a variable in the denominator.

There is a very important implication when a variable is in the denominator of an expression or equation. Dividing any quantity by zero leads to an undefined value, so whenever the denominator is equal to zero, the equation is undefined. Sometimes, when solving rational equations, we can make all the right moves algebraically, but the solution we get for our variable makes our denominator equal to zero. These solutions are known as extraneous solutions.

HINT

When solving rational equations, always check to make sure the solution does not make the denominator equal zero. If it does, the solution is extraneous, and should not be included in your final answer as a solution to the rational equation.

A useful strategy in solving rational equations is to rewrite the equation so that every term has a common denominator. The reason for this is that once all terms are written to have a common denominator, we can create an equation with just the numerators.

EXAMPLE Find the solution to $\frac{2x}{x+1}+\frac{3}{x+1}=\frac{x-4}{x+1}$.

Since all the fractions have the same denominator, we can just focus on the numerators.

$\frac{2x}{x+1}+\frac{3}{x+1}=\frac{x-4}{x+1}$	Use numerators only
$2x+3=x-4$	Subtract 3 from both side
$2x=x-7$	Subtract x from both sides
$x=-7$	Our solution

Because each term in the rational equation has the same expression for its denominator, this also represents a solution to the rational equation. The only thing we need to do is check to make sure that when $x=-7$, we never have a denominator of zero in any of the terms. Since the denominator is $x+1$, and $-7+1$ is equal to -6, which is non-zero, we can say that $x=-7$ is the solution to this rational equation.

TERM TO KNOW

Rational Equation

An equation with a least one rational expression.

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2. Finding the Least Common Denominator

Now that we have a useful strategy for solving rational functions, let's talk about the processes for creating terms with common denominators. The easiest method to find a common denominator between rational expressions is to multiply the terms in the denominator together.

EXAMPLE Find a common denominator between these algebraic fractions:

$$\frac{1}{x^2+2} \qquad \frac{x+3}{x} \qquad \frac{x}{5}$$

To find a common denominator, we would multiply each denominator, x^2+2, x, and 5, together.

$$(x^2+2)\cdot x\cdot 5=(x^2+2)\cdot 5x=x^2\cdot 5x+2\cdot 5x=5x^3+10x$$

The common denominator is $5x^3+10x$.

This method works for all cases, but it can get a bit messy sometimes with algebraic fractions. One strategy is to find the lowest common denominator by factoring each expression and multiply all of the distinct factors.

EXAMPLE Find the least common denominator between these algebraic fractions:

$$\frac{4}{x(x+2)} \qquad \frac{x-8}{3x} \qquad \frac{x}{x+2}$$

First, factor each denominator as much as you can:

$$\frac{4}{x(x+2)} \qquad \frac{x-8}{3\cdot x} \qquad \frac{x}{(x+2)}$$

Now that we've factored each denominator as much as we can, we're going to look at the greatest number of times each factor is used. Let's first look at *x*, which is used once, so we're going to include that as part of the lowest common denominator:

Next, look at $(x+2)$. This is used once in the first fraction and once in the third fraction, so the greatest number of times it is used is once. So $(x+2)$ will also be included in the lowest common denominator:

The next factor is 3, which is used at most once in any fraction, so we will include a 3 in the lowest common denominator:

$$\text{lowest common denominator} = x \cdot (x+2) \cdot 3$$

We can rewrite this as:

$$\text{lowest common denominator} = 3x(x+2)$$

Once we have our common denominator, we can solve for equation by following these steps:

1. Multiply both the numerator and the denominator of the first fraction by the denominator of both the second fraction and third fraction.
2. Repeat the process with the second fraction: multiply both the numerator and the denominator of the second fraction by the denominator of the first fraction and third fraction.
3. Repeat for the third fraction; multiply both the numerator and the denominator of the third fraction by the denominator of the first fraction and second fraction.
4. Add or subtract numerators and keep the denominator the same.

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3. Solving Rational Equations using a Common Denominator

Let's solve the following rational equation using our common denominator strategy:

↪ EXAMPLE Find the solution to $\frac{1}{x-4} + \frac{5}{x} = \frac{4}{3}$.

First, we need to find a common denominator. Luckily for us, each denominator is already factorized, so the least common factor is the product of the 3 denominators:

$$(x-4)(x)(3) = 3x^2 - 12x$$

The tricky part is now adjusting our numerators to the original fractions, in order to be equivalent with a new denominator. To do this, let's express each part of the fraction in factored form. This will help identify which factors need to be attached to the numerator:

$\frac{1}{x-4}$	The denominator needs factors (3) and (x) ; multiply both numerator and denominator
$\frac{1(3)(x)}{(x-4)(3)(x)}$	Evaluate multiplication
$\frac{3x}{3x^2-12x}$	Equivalent fraction to $\frac{1}{x-4}$

$\frac{5}{x}$	The denominator needs factors (3) and $(x-4)$; multiply both numerator and denominator
$\frac{5(3)(x-4)}{x(3)(x-4)}$	Evaluate multiplication
$\frac{15x-60}{3x^2-12x}$	Equivalent fraction to $\frac{5}{x}$

$\frac{4}{3}$	The denominator needs factors (x) and $(x-4)$; multiply both numerator and denominator
$\frac{4(x)(x-4)}{3(x)(x-4)}$	Evaluate multiplication
$\frac{4x^2-16x}{3x^2-12x}$	Equivalent fraction to $\frac{4}{3}$

Notice how we multiplied each fraction by the other denominators. Now we can ignore the denominator (the expression we worked so hard to find!) and create an equation with no denominator at all:

$\frac{1}{x-4}+\frac{5}{x}=\frac{4}{3}$	Multiply each fraction by the other denominators
$\frac{1(3)(x)}{(x-4)(3)(x)}+\frac{5(3)(x-4)}{x(3)(x-4)}=\frac{4(x)(x-4)}{3(x)(x-4)}$	Evaluate multiplication
$\frac{3x}{3x^2-12x}+\frac{15x-60}{3x^2-12x}=\frac{4x^2-16x}{3x^2-12x}$	Create equation with just numerators; ignore denominators
$3x+(15x-60)=4x^2-16x$	Equation with just numerators

This is actually a quadratic equation we can solve using the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, once we set one side of the equation equal to zero:

$3x+(15x-60)=4x^2-16x$	Combine $3x$ and $15x$ on the left side
$18x-60=4x^2-16x$	Subtract $18x$ and add 60 to both sides
$0=4x^2-16x-18x+60$	Combine -16x and -18x on the right side
$0=4x^2-34x+60$	Identify coefficients a, b, and c
$a=4,\ b=-34,\ c=60$	Plug these values into quadratic formula
$x=\frac{-(-34)\pm\sqrt{(-34)^2-4(4)(60)}}{2(4)}$	Evaluate

$x = 2.5, x = 6$ Our solutions

Be sure to plug in 2.5 and 6 into all denominators from our original rational equation. If either 2.5 or 6 makes one of the denominators equal to zero, we have an extraneous solution.

$\frac{1}{x-4} + \frac{5}{x} = \frac{4}{3}$	Plug in 2.5 for *x*
$\frac{1}{2.5-4} + \frac{5}{2.5} = \frac{4}{3}$	Evaluate
$\frac{1}{-1.5} + \frac{5}{2.5} = \frac{4}{3}$	Plugging 2.5 in for *x* does NOT result in any denominator of zero, so this is NOT an extraneous solution

$\frac{1}{x-4} + \frac{5}{x} = \frac{4}{3}$	Plug in 6 for *x*
$\frac{1}{6-4} + \frac{5}{6} = \frac{4}{3}$	Evaluate
$\frac{1}{2} + \frac{5}{6} = \frac{4}{3}$	Plugging 6 in for *x* does NOT result in any denominator of zero, so this is NOT an extraneous solution

Since no denominator equals 0 at $x = 2.5$ or $x = 6$, there are no extraneous solutions to the rational equation.

SUMMARY

There are some cases where **rational equations have extraneous solutions**. If a potential solution makes the denominator of a rational expression equal to 0, then the solution is extraneous. We can solve a rational equation by **finding the least common denominator**. The easiest method is to multiply all the denominators together. Then, because the denominators are the same, you can write and solve a rational equation using just the numerators of each fraction.

TERMS TO KNOW

Rational Equation
An equation with at least one rational expression.